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DYNAMICS OF A SYSTEM OF REMOTE PUNCHES ON AN ELASTIC HALF-SPACE

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The dynamic interaction (contact) between an elastic half-space and several smooth punches is studied. It is assumed that the dimensions of the contact regions Ω_i are much smaller than the distances between them and the scale of time of the process considered is comparable with the time required for an elastic wave to travel from one region to another. An asymptotic approach to the solution of the problem is proposed and the first two terms of the asymptotic representation of the displacement in the contact region and its neighborhood are constructed.

Introduction. The numerical solution of the given dynamic three-dimensional problem by the finite-element, boundary-element, and other methods generally involves the principal difficulty associated with degeneration of the contact region into a set of points. Therefore, to solve the problem, we use asymptotic methods.

The integral equation of the problem is constructed using the closed form of the fundamental solution of the nonstationary half-space problem (Lamb's problem) [1]. The asymptotic simplification of this equation yields a problem of smaller dimensionality that is solved numerically.

As an example, we consider the vertical motion of several smooth round punches with a plane base. Some results have already been reported at conferences [2–4].

The asymptotic approach to the corresponding static problem was used for the first time by Galin [5]. Argatov and Nazarov [6] obtained a rigorous asymptotic solution for static loading of an elastic body resting on several small supports. In the 1970s, attempts were undertaken to solve dynamic problems with a small parameter, but these solutions contained assumptions [7–9]. The bibliography of studies devoted to numerical solutions can be found in [10–16].

1. Fundamental Solution of Lamb's Problem. We use the closed form of the fundamental solution $G(t,r)$ of the nonstationary Lamb's problem [1]. Being caused by a vertical force suddenly applied to the boundary at the moment $t = 0$, whose magnitude does not change at subsequent times, the vertical displacement of the points at the plane boundary $z = 0$ of the elastic half-space $z \leqslant 0$ has the form

$$G(t,r) = \frac{1-\nu}{2\pi\mu r}\Big[\frac{1}{2}H\Big(t-\frac{r}{c_1}\Big)+\frac{1}{2}H\Big(t-\frac{r}{c_2}\Big)+q\frac{c_2 t}{r}H\Big(t-\frac{r}{c_1}\Big)$$

$$\times H\Big(\frac{r}{c_2}-t\Big)-2q_3\Big(\gamma^2-\frac{{c_2}^2t^2}{r^2}\Big)^{-1/2}H\Big(t-\frac{r}{c_2}\Big)H\Big(\frac{r}{c_{\rm R}}-t\Big)\Big], \qquad (1)$$

$$q(s) = -q_1(s^2-p_1)^{-1/2}+q_2(s^2-p_2)^{-1/2}-q_3(\gamma^2-s^2)^{-1/2},$$

for the value of Poisson's ratio $\nu < 0.263$, where μ is the shear modulus, r is the distance between the observer's point and the point at which the force is applied, c_1 and c_2 are the velocities of the extension and shear waves, respectively, $c_{\rm R}$ is the velocity of the Rayleigh wave, $H(t)$ is the Heaviside function, and

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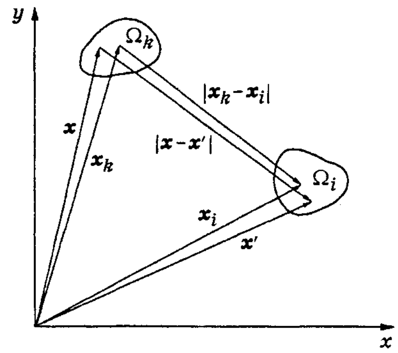

Fig. 1. Location of two neighboring punches at the half-space surface $z = 0$.

$\gamma = c_2/c_R$, $q_n = q_n(\nu)$, and $p_m = p_m(\nu)$ are dimensionless constants. For $\nu = 1/4$, we have $q_1 = \sqrt{3}/12$, $q_2 = \sqrt{3\sqrt{3} - 5/12}$, $q_3 = \sqrt{3\sqrt{3} + 5/12}$, $p_1 = 1/4$, $p_2 = (3 - \sqrt{3})/4$, $\gamma = \sqrt{3 + \sqrt{3}/2}$, and $\alpha = c_2/c_1 = 1/\sqrt{3}$. The numerical results given in this paper were obtained for these values of the coefficients.

2. Construction of the Asymptotic Representation for the Displacement. We consider a system of N round smooth (friction is absent) punches located at the boundary $z = 0$ of the elastic half-space $z \leqslant 0$ [the contact region is $\Omega = \cup\Omega_i$ $(i = 1, \ldots, N)$, where Ω_i are the contact zones] (Fig. 1). The seismic pulse of the displacement $w_0 = w_0(t/T)$ of the boundary $z = 0$ causes the vertical displacement of the punches (T is the scale of time). The function w_0 is a sufficiently smooth bounded finite function (or the sum of the finite and Heaviside functions). The maximum radius of the punch, $h = \max\limits_k\{h_k\}$, is assumed to be much smaller than the minimum distance $l = \min\limits_{i,k}\{|\boldsymbol{x}_i - \boldsymbol{x}_k|\}$ between their centers [$\varepsilon = h/l = o(1)$, and $\boldsymbol{x}_i$ is the radius-vector of the center of the ith punch]. The scale of time T of the external action w_0 (and, hence, the process) is comparable with the time l/c_2 required for the shear wave to travel between the nearest punches [$c_2T/l = O(1)$].

Using the fundamental equation (1), we write the following boundary integral equation of elastodynamics:

$$w - w_0 = \sum_{i=1}^{N} \int_0^t \iint_{\Omega_i} \frac{\partial}{\partial(t-t')} G(t-t', |\boldsymbol{x} - \boldsymbol{x}'|)\sigma\Big(\frac{t'}{T}, \boldsymbol{x}'\Big)\, dt'\, d\Omega(\boldsymbol{x}'), \tag{2}$$

where $w(\boldsymbol{x}, t/T)$ is the vertical displacement and $\sigma(\boldsymbol{x}, t/T)$ is the contact pressure.

The purpose of an asymptotic analysis is to simplify equality (2) with the use of the adopted assumptions on the character of the process and on the fact that the punches are remote from each other. To estimate the deflection w at the points of the region $\boldsymbol{x} \in \Omega_k$, we rewrite (2), separating the contribution of the contact stresses in this region:

$$\begin{aligned} w - w_0 = &\int_0^t \iint_{\Omega_k} \frac{\partial}{\partial(t-t')} G(t-t', |\boldsymbol{x} - \boldsymbol{x}'|)\sigma\Big(\frac{t'}{T}, \boldsymbol{x}'\Big)\, dt'\, d\Omega(\boldsymbol{x}') \\ &+ \sum\nolimits' \int_0^t \iint_{\Omega_i} \frac{\partial}{\partial(t-t')} G(t-t', |\boldsymbol{x} - \boldsymbol{x}'|)\sigma\Big(\frac{t'}{T}, \boldsymbol{x}'\Big)\, dt'\, d\Omega(\boldsymbol{x}'). \end{aligned} \tag{3}$$

Here the notation $\sum' = \sum\limits_{i=1,\, i\neq k}^{N}$ is introduced.

The small dimensions of the region Ω_i in comparison with the distances to the neighboring regions enables one to estimate asymptotically the contribution from the terms corresponding to load convolutions with the fundamental solution over the regions Ω_i $(i \neq k)$ to the displacement. The distances $|\boldsymbol{x} - \boldsymbol{x}'|$ and $|\boldsymbol{x}_k - \boldsymbol{x}_i|$ differ slightly, i.e., the quantity $(|\boldsymbol{x}_k - \boldsymbol{x}_i| - |\boldsymbol{x} - \boldsymbol{x}'|)/|\boldsymbol{x}_k - \boldsymbol{x}_i|$ is a quantity of the order $O(\varepsilon)$ (Fig. 1). The Green's function $G(t, r)$ depends on the distance between the observer's point and the point where the force is applied; therefore, representing $G(t,r)$ as the sum $G(t, |\boldsymbol{x}_k - \boldsymbol{x}_i|) + [G(t,r) - G(t, |\boldsymbol{x}_k - \boldsymbol{x}_i|)]$, one can find a term with the factor $G(t, |\boldsymbol{x}_k - \boldsymbol{x}_i|)$ that is coordinate-independent and show that the other part is a higher-order quantity. To this end, after the integral is taken over time by parts, we use the series expansion in the small parameter $\varepsilon = \max\limits_k\{h_k\}/\min\limits_{i,k}\{|\boldsymbol{x}_i - \boldsymbol{x}_k|\}$ in the second term on the right side of Eq. (3). After transformations, dividing both sides of the equation by h, we obtain

$$\frac{w}{h} - \frac{w_0}{h} \sim \frac{1}{h} \iint\limits_{\Omega_k} \Big(\frac{\partial}{\partial t} G(t, |\boldsymbol{x} - \boldsymbol{x}'|)\Big) * \sigma\Big(\frac{t}{T}, \boldsymbol{x}'\Big)\, d\Omega(\boldsymbol{x}')$$

$$+\varepsilon \sum{}' \Big[\mu l\, G\Big(\frac{t}{T}, \frac{|\boldsymbol{x}_k - \boldsymbol{x}_i|}{bl}\Big)\Big] * \frac{\partial}{\partial(t/T)} \Big(\frac{F_i(t/T)}{\mu h^2}\Big), \tag{4}$$

$$\varepsilon \to 0, \qquad F_i\Big(\frac{t}{T}\Big) = \iint\limits_{\Omega_i} \sigma\Big(\frac{t}{T}, \boldsymbol{x}'\Big)\, d\Omega(\boldsymbol{x}'),$$

where the asterisk denotes convolution in time t/T and $b = (Tc_2)/l$.

The first term on the right side of Eq. (4) can be simplified using the assumption on slow variation of the pressure σ in the scale of time required for the shear wave to travel through the region Ω_k. In this case, the small parameter is the ratio $h/(Tc_2) = \varepsilon/b$. We integrate this term over time by parts. Representing the Green function in the form $G(t, |\boldsymbol{x} - \boldsymbol{x}'|) = G^3(|\boldsymbol{x} - \boldsymbol{x}'|)H(t) + [G(t, |\boldsymbol{x} - \boldsymbol{x}'|) - G^3(|\boldsymbol{x} - \boldsymbol{x}'|)H(t)]$, performing asymptotic expansions, and integrating over time, we obtain

$$\frac{w}{h} - \frac{w_0}{h} \sim \iint\limits_{\Omega_k} [\mu h\, G^3(|\boldsymbol{x} - \boldsymbol{x}'|)]\, \frac{\sigma(t/T, \boldsymbol{x}')}{\mu}\, \frac{d\Omega(\boldsymbol{x}')}{h^2}$$

$$+\varepsilon \sum{}' \Big[\mu l\, G\Big(\frac{t}{T}, \frac{|\boldsymbol{x}_k - \boldsymbol{x}_i|}{bl}\Big)\Big] * \frac{\partial}{\partial(t/T)} \Big(\frac{F_i(t/T)}{\mu h^2}\Big) + \varepsilon\, \frac{(1-\nu)A}{2\pi b}\, \frac{\partial}{\partial(t/T)} \Big(\frac{F_k(t/T)}{\mu h^2}\Big), \tag{5}$$

$$\varepsilon \to 0, \qquad A = A(\nu) = -\frac{1}{2} - \frac{1}{2}\frac{c_2}{c_1} + \int\limits_\alpha^1 q(s)\, ds - 2q_3 \arccos \frac{1}{\gamma} < 0,$$

where $G^3 = (1-\nu)/(2\pi\mu r)$ is the Boussinesq solution [17].

We now consider small rotations of the punch about the horizontal axes x and y. An asymptotic analysis of the contact equation (2) shows that the contribution of the terms connected with rotation to the displacement is of the order $O(\varepsilon^2)$. Therefore, in determining the deflection with accuracy up to $O(\varepsilon^2)$, rotations can be ignored.

In this case, where the operator of convolution over the region Ω_k on the right side of equality (5) admits inversion, one can continue analysis of this equality, using the known solution of the corresponding static problem and treating (5) as an equation for a desired σ. For the displacement w_k of the kth punch, we obtain

$$\frac{w_k}{h} - \frac{w_0}{h} \sim \frac{1-\nu}{4}\, \frac{h}{h_k}\, \frac{F_k(t/T)}{\mu h^2} + \varepsilon\, \frac{(1-\nu)A}{2\pi b}\, \frac{\partial}{\partial(t/T)} \Big(\frac{F_k(t/T)}{\mu h^2}\Big)$$

$$+\varepsilon \sum{}' \Big[\mu l\, G\Big(\frac{t}{T}, \frac{|\boldsymbol{x}_k - \boldsymbol{x}_i|}{bl}\Big)\Big] * \frac{\partial}{\partial(t/T)} \Big(\frac{F_i(t/T)}{\mu h^2}\Big), \quad \varepsilon \to 0. \tag{6}$$

(For an arbitrary shape of the region Ω_k, this inversion can easily be performed numerically.)

At first sight, the resulting contact equation (6) distorts the physical nature of the process. Indeed, considering it as an equation for F_k, one can construct a formal solution that contains the exponentially increasing term (as $t \to \infty$). However, more careful analysis shows that the bounded solution can be constructed with an appropriate choice of the constant (having the meaning of the initial condition) in the solution. The initial phase of the dynamic process is described with distortions. [The real wave character of the initial phase of the process can be described only by the exact equation (2).] The long-wave asymptotic model (6) describes neither the beginning of the process (the duration is of the order h/c_2) nor the changes that occur in the system for the time h/c_2.

To close the dynamic formulation of the problem, we use the second Newton law. For the kth punch, we have

$$M_k \frac{d^2}{dt^2} w_k\Big(\frac{t}{T}\Big) = -F_k\Big(\frac{t}{T}\Big),$$

where M_k is the mass of the kth punch.

We introduce the dimensionless time $\tau = t/T$, the dimensionless parameters of the order $O(1)$, namely, $b = Tc_2/l$, $\delta_i = M_i/M$, and $\theta_i = h_i/h$, $\beta = M/(l^2 h\rho)$, $æ_i = (2\theta_i/(\pi b))|A|$, $\alpha_i = \theta_i^2 b^2/(\delta_i\beta)$, and the dimensionless functions $g_{ik}(\tau) = G(|\boldsymbol{x}_i - \boldsymbol{x}_k|, t)\mu l$, $v_i(\tau) = w_i(t/T)/h$, $v_0(\tau) = w_0(t/T)/h$, and $f_i(\tau) = F_i(t/T)(1/(h_i^2\mu))$, where $M = \min\limits_i\{M_i\}$ and ρ is the density of the medium. (In principle, the approach proposed in this paper makes it possible to analyze a broader range of variation of the parameters β and α_i.)

We rewrite the system in dimensionless form:

$$v_k(\tau) - v_0(\tau) = \frac{1-\nu}{4}\,\theta_k f_k(\tau) - \varepsilon\,\frac{1-\nu}{4}\,\theta_k æ_k \dot f_k(\tau) + \varepsilon{\sum}'\theta_i^2 g_{ik}(\tau) * \dot f_i(\tau),$$

$$\ddot v_k(\tau) = -\alpha_k f_k(\tau) \qquad (k = 1,\dots,N).$$

The dots denote differentiation with respect to the dimensionless time τ and the asterisk denotes convolution in τ. The system is to be supplemented by the initial conditions $v_k(0)$ and $\dot v_k(0)$ $(k = 1,\dots,N)$.

We shall seek a bounded [$\lim v_k(\tau) < \infty$ and $\lim f_k(\tau) < \infty$ as $\tau \to \infty$] solution of the problem. Using the asymptotic estimation $f_k(\tau) - \varepsilon æ_k \dot f_k(\tau) = f_k(\tau - \varepsilon æ_k) + O(\varepsilon^2)$, we obtain $v_k(\tau) - v_0(\tau) - \varepsilon{\sum}'\theta_i^2\, g_{ik}(\tau) * \dot f_i(\tau) = ((1-\nu)/4)\theta_k f_k(\tau - \varepsilon æ_k) + O(\varepsilon^2)$. Regarding this equality as a functional equation for the desired function $f_k(\tau)$ and solving it, we arrive at the contact equation

$$\frac{1-\nu}{4}\,\theta_k f_k(\tau) = v_k(\tau + \varepsilon æ_k) - v_0(\tau + \varepsilon æ_k) - \varepsilon{\sum}'\theta_i^2 \int\limits_0^{\tau+\varepsilon æ_k} g_{ik}(\tau + \varepsilon æ_k - s)\dot f_i(s)\,ds + O(\varepsilon^2). \qquad (7)$$

Expanding the right side of (7) into a series in terms of the small parameter ε, we finally obtain the system of equations of the problem in the form

$$\begin{gathered}\frac{1-\nu}{4}\,\theta_k f_k(\tau) = v_k(\tau) + \varepsilon æ_k \dot v_k(\tau) - v_0(\tau + \varepsilon æ_k) - \varepsilon{\sum}'\theta_i^2\, g_{ik}(\tau) * \dot f_i(\tau) + O(\varepsilon^2),\\ \ddot v_k(\tau) = -\alpha_k f_k(\tau).\end{gathered} \qquad (8)$$

3. Dynamics of the System of Round Punches. Calculations were performed for two ($N = 2$) equal punches $v_1 = v_2 = v$ under the initial conditions $v(0) = \dot v(0) = 0$. System (8) was solved numerically for various values of the parameters ε (for $\beta = 1.5$) (Fig. 2) and β (for $\varepsilon = 0.1$) (Figs. 3–5). Poisson's ratio was taken to be $\nu = 1/4$ and $b = 1$. The seismic pulse was taken in the form $w_0 = \sin^2(\pi t/T)H(t)H(1 - t/T)$. Figures 2, 3, and 5 show the dimensionless displacement of the punch v versus time, and Figs. 4 and 6 show the contact force f under the punch versus time.

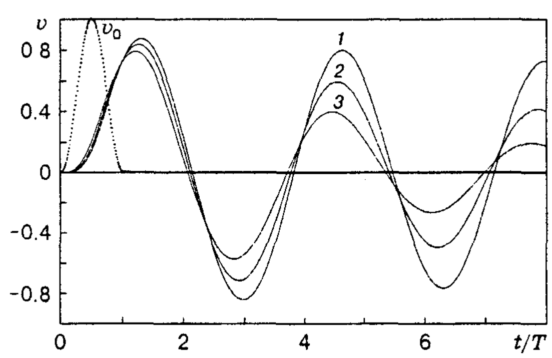

Fig. 2

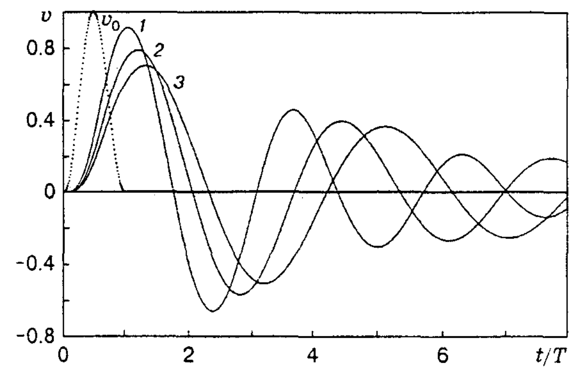

Fig. 3

Fig. 2. The displacement v versus time ($\beta = 1.5$): $\varepsilon = 0.01$ (1), 0.05 (2), and 0.1 (3).

Fig. 3. The displacement v versus time ($\varepsilon = 0.1$): $\beta = 1$ (1), 1.5 (2), and 2 (3).

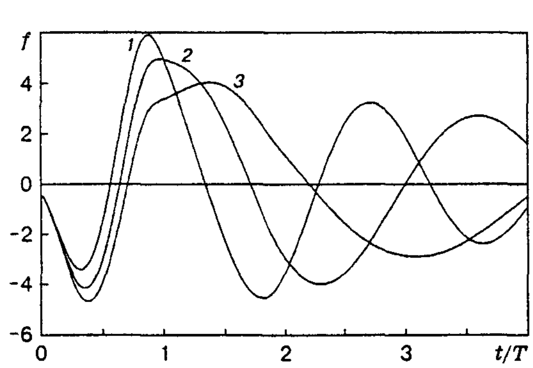

Fig. 4

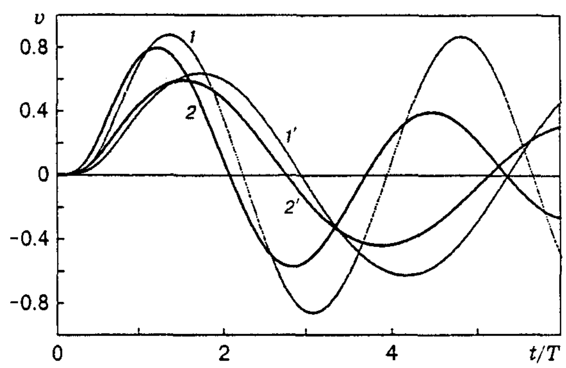

Fig. 5

Fig. 4. The reaction f of the elastic medium versus time ($\varepsilon = 0.1$): $\beta = 0.5$ (1), 1 (2), and 2 (3).

Fig. 5. The displacement v versus time ($\varepsilon = 0.1$): quasistatic model (curves 1 and 1′); dynamic model (curves 2 and 2′); $\beta = 1.5$ (curves 1 and 2) and 3 (curves 1′ and 2′).

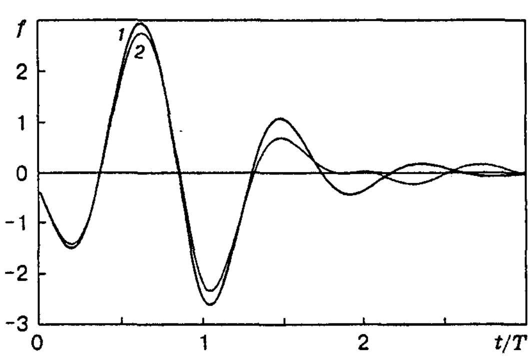

Fig. 6. The reaction f of the elastic medium versus time ($\beta = 0.1$ and $\varepsilon = 0.1$): curve 1 refers to the isolated punch and curve 2 to the system of two punches.

Figure 5 show the displacement v versus time that was calculated by the dynamic and quasistatic models. The latter model is described by the equation

$$\beta\Big(1 + \frac{2}{\pi}\varepsilon\Big)\frac{1-\nu}{4}\ddot{v} + v = v_0.$$

It is clear from Fig. 5 that the quasistatic model overestimates the displacement compared to the dynamic model (8). Figure 6 shows contact forces acting under the isolated punch (curve 1) and in the case of a system of two punches (curve 2) that were calculated by the proposed asymptotic model. It follows from Fig. 6 that the contact forces cannot be determined correctly if the influence of the second punch is ignored.

Conclusions. The advantages of the resulting contact equation are as follows: its dimensionality is smaller compared to that of the initial integral equation of elastodynamics and it is suitable for numerical solution. The proposed approach can be generalized to the following cases:

— dynamic interaction between the half-space and the bodies occurs over the three-dimensional contact region (under the condition that the depth of their penetration into the medium is much smaller than the distance between them);

— the displacement and stress at the boundary have three components;

— the bodies interacting with a half-space are deformable;

— there is a small plastic zone in the contact region.

It should be noted that if the scale of time of the process far exceeds the maximum time required for the compression wave to travel between the regions Ω_i, the dynamic problem becomes a quasistatic problem (with parameter t). If the time scale is comparable with the time required for the shear wave to travel between the regions, the proposed asymptotic approach is effective. In the case of a short-duration process, where its scale is comparable with the time required for the elastic wave to run across the region Ω_i, the interaction between the punches through the medium can be ignored and the motion can be calculated by solving the dynamic problem for an isolated punch. Finally, if this scale is much smaller than the time required for the wave to travel across the region, the problem for a plane punch becomes almost one-dimensional, since the mutual influence at different points in the contact region does not have time to show up.

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